Theorem

Partitioning: Adding numbers, then counting them.

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MATH 4620: Combinatorics

Today's Overview

Introduction:

- What are Partition Numbers?
- Diagram Representations
- Order Matters

Unordered Partition Numbers:

- History and Discovery
- Euler
- Ramanujan
- Ono

and
lastly, we will end off with some applications.

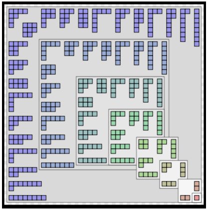

Figure: Young diagrams associated to the partitions of the positive integers 1 through 8

What are Partition Numbers?

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$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, ...$$

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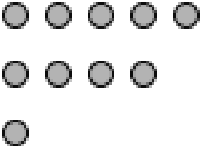
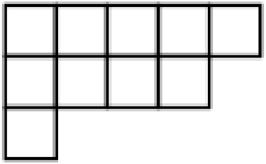
- Number of ways to break down a number into positive parts
- Leads to the most fascinating areas of **Number Theory**, **Analysis**, **Algebraic Geometry, Chaos Theory**
- And of course; **Combinatorics**.

Diagrams Representation

Ferrers diagram vs. Young diagram

$$5 + 4 + 1$$

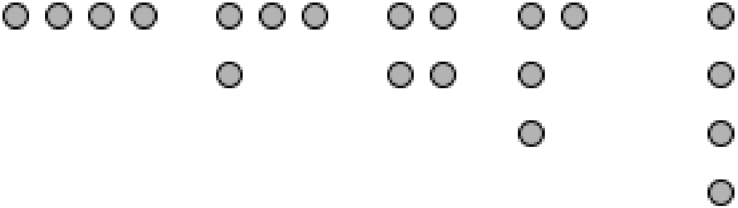




$$= 10$$

Diagrams Representation

$$p(n) = 5$$



$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

Number of Ways to Partition Numbers
Unordered vs. Ordered

Unordered:

$$4 = 3 + 1 = 2 + 2 = 1 + 2 + 1 = 1 + 1 + 1 + 1$$

$$p_u(4) = 5$$

Ordered:

$$4 = 1{+}1{+}1{+}1 = 2{+}2 = 1{+}1{+}2 = 1{+}2{+}1 = 2{+}1{+}1 = 1{+}3 = 3{+}1$$

$$p_o(4) = 8$$

Number of Ways to Partition Numbers
Order Matters

If we were to just test out numbers and count by hand, for the first few we would find the sequence (starting at n=1):

$$1, 2, 4, 8, 16, 32, ...$$

So it seems like:

$$p_o(n) = 2^{n-1}$$

We can prove this pretty easily.

Number of Ways to Partition Numbers
Order Matters

Suppose n is entirely made up of n 1's

$$n : 1(\quad)1(\quad)1(\quad)\ldots(\quad)1$$

For each space, we can either partition n with a plus sign or nothing. Nothing would represent that the two 1's beside a space are added together to give a larger value, where a plus means there is a partition. So there are $n-1$ spaces with a choice of two objects (partition or add adjacent numbers) therefore we get a total number of 2^{n-1} ways to partition the number n.

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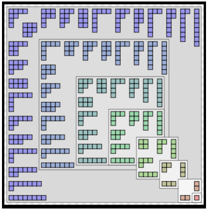


Figure: Young diagrams associated to the partitions of the positive integers 1 through 8

Theorem

The generating function for the partition function $p(n)$ is

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty}(1-x^k)^{-1}$$

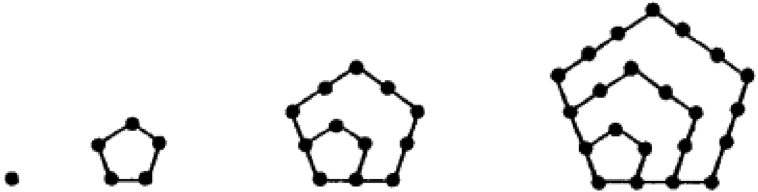
Theorem

$$\prod_{k=1}^{\infty}(1-x^k) = 1 + \sum_{m=1}^{\infty}(-1)^m \left(x^{\omega(m)} + x^{\omega(-m)}\right)$$

History and Discovery
Euler (1707-1873)

- Pentagonal Numbers: $\frac{3m^2-m}{2}$ for m integer.
- Generalized: 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77,...
- $p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\dots$
- $p(n) = \sum_k (-1)^{k-1} p(n-g_k)$

where g_k is the kth generalized pentagonal number. Also note that $p(n) = 0$ when n is negative.



$1 \qquad 1 + 4 = 5 \qquad 1 + 4 + 7 = 12 \qquad 1 + 4 + 7 + 10 = 22$

Hardy and Ramanujan found this asymptotic expression

$$p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \ as \ n \to \infty$$

A bit later, they improved their approximation to:

$$p(n) \approx \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{v} A_k(n)\sqrt{k} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}} e^{\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})}}\right)$$

where

$$A_k(n) = \sum_{0 \le m < k, (m,k)=1} e^{\pi i (s(m,k) - 2nm/k)}$$

where $s(m,k)$ is a Dedekind sum and (m,k) in the summation implies that the sum only occurs for values over the values of m that are relatively prime to k.

History and Discovery
Ramanujan (1886-1969)

Ramanujan also discovered that congruences in the number of partitions exists for numbers ending in 4 and 9:

$$p(5k + 4) \equiv 0 \ (mod\ 5)$$

and later also found:

$$p(7k + 5) \equiv 0 \ (mod\ 7)$$

$$p(11k + 6) \equiv 0 \ (mod\ 11)$$

The next one in the pattern is:

$$p(11^3 \cdot 13 \cdot k + 237) \equiv 0 \ (mod\ 13)$$

found by AOL Atkin. And found by Hans Rademacher found this convergent series for $p(n)$

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \cdot \frac{d}{dn}\left(\frac{1}{\sqrt{n-\frac{1}{24}}} \sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}(n-\frac{1}{24})}\right]\right)$$

Thanks to the work of Euler, Ramanujan, Atkin and Rademacher, Ken Ono and his collabators were able to find structure that would add a whole new dimension to how the problem of partitions are being handled.

History and Discovery
Ken Ono (1969-2028)

In 2000:

$$p(h - p_i n) \equiv 0 \ (mod \ p_i)$$

where p_i is prime
and in 2001:

$$p(h - kn) \equiv 0 \ (mod \ k)$$

k not divisible by 3
and finally, in 2011, Ken Ono and his collabarators paper; ℓ-ADIC PROPERTIES OF THE PARTITION FUNCTION, found self-similarity structure in the congruences of the partition number.

History and Discovery
Ken Ono (1969-2031)

- Sequences mod ℓ^m are eventually **periodic** (i.e self-similar).
- periods are **the same for all** n
- increasing m as one zooms gives **increased resolution** without impacting **periodicity**
- There is a **Hausdorff dimension** for each $\ell \geq 5$
- The Hausdorff dimension for 5,7 and 11 is zero.

Theorem (Folsom, Kent, O)

For primes $\ell \geq 5$, the partition numbers are ℓ-adically fractal with bounded Hausdorff dimensions.

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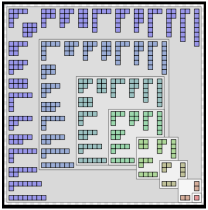

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Partition Numbers: Applications

Partitions are used in the mathematics of shuffling, which is useful for many physical phenomena and areas of mathematics involving direct use of partitions.

- Physics: statistical mechanics, for the notion of energy: we can shuffle around and partition for how energy transforms from one form to another since

$$K + P = E_{tot}$$

- How many shuffles will give total 'randomness'?
- Genetics: Ewen's Sampling formula
- Probability theory: Chinese restaurant process

Proof of Slide 8 Theorems

Theorem

Generating Function $P(x)$ for partition function $p(n)$ is

$$P(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-1}$$

Proof.

Partition of $n = \sum_{i=1}^{m} ir_i$ with r_i terms equal to i, for $1 \leq i \leq m$, corresponds to the term $x_1^{r_1} \cdots x_m^{r_m}$ in the infinite product $\prod_{k=1}^{\infty}(1 + x_k + x_k^2 + \cdots + x_k^n)$. The series $(1 + x_k + x_k^2 + \cdots + x_k^n)$ is equal to $\frac{1}{1-x_k}$ and letting $x_k = x^k$, we get the result

$$\prod_{k=1}^{\infty} (1 - x^k)^{-1}$$

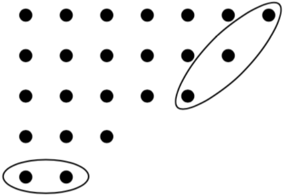
$\square$

Consider $P(x)^{-1} = \prod_{k=1}^{\infty}(1 - x^k)$. The expansion of this product, a partition of n into unequal parts contributes to $+1$ to the coefficient of x^n if the number of parts is even and -1 if the number of parts is odd. So the coefficient of x^n is $p_e(n) - p_o(n)$, where $p_e(n)$ is the partitions of n into an even number of unequal parts and $p_o(n)$ is the partition of n into odd number of unequal parts. Euler proved that $p_e(n) = p_o(n)$ unless n was a pentagonal number. We denote the pentagonal numbers as $\omega(m)$.

Theorem

$$\prod_{k=1}^{\infty}(1-x^k) = 1 + \sum_{m=1}^{\infty}(-1)^m\left(x^{\omega(m)} + x^{\omega(-m)}\right)$$

Proof: Consider n partitioned into an unequal amount of parts. Using 23 as an example, $23 = 7 + 6 + 5 + 3 + 2$. We have the following Ferrers diagram:



The number of particles in last row is denoted as b (for base) and the number of particles in the diagonal is denoted as s (for slope). For our example we have $b = 2$ and $s = 3$. Defining two new operations, **A** and **B** as:

A: If $b \leq s$ then remove the base and adjoin it to the diagram at the right to form a new slope parallel to the original one, unless when $b = s$ which then the base and slope have a point in common. This case creates an exception that can only occur if $n = b + (b + 1) + \cdots + (2b - 1) = \omega(b)$.

B: If $b > s$ then remove the slope and adjoin it at the bottom of the diagram a sa new base, except for the case of $b = s + 1$ and the base and slope have a point in common, which then makes an exceptional case that can only occur if $n = (s + 1) + \cdots + 2s = \omega(-s)$.

In all other cases exactly one of the operations can be carried out which makes a one-to-one correspondence between the partitions of n into an even number of unequal parts and partitions of n into an odd number of unequal parts. Hence for values of n we have $p_e(n) - p_o(n) = 0$. The exceptional cases the difference is $+1$ or -1. **Q.E.D**